

Finding Plasma Equilibria with Magnetic Islands

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The traditional method of solving the helically symmetric plasma equilibrium equation, of the form $L[G] = F(G, r)$ where L is an elliptic linear operator, has been the simple iteration $L[G^{n+1}] = F(G^n, r)$. A model of a Tokamak equilibrium is constructed and used to illustrate the divergence of the simple iteration for plasma equilibria with magnetic islands. Although the problem of equilibria with magnetic islands is two dimensional, for small islands the numerical stability of the simple iteration may be analyzed using a one-dimensional equation similar to the linearized equilibrium equation used to analyze physical (resistive) instability. This analysis is used to prove that any equilibria of the Tokamak type with small islands cannot be obtained by the simple iteration and to illustrate the superlinear convergence of Newton's method on these problems. The implementation of Newton's method is discussed and examples are given. © 1988 Academic Press, Inc.

1. INTRODUCTION

In this paper we consider methods for solving the equation describing mechanical equilibrium of a plasma in a magnetic field, in situations with helical symmetry having dependence on only two variables r and $u \equiv m\theta + kz$, where r, θ, z form a cylindrical coordinate system. Defining

$$G \equiv krA_\theta - mA_z$$

and

$$H \equiv krB_\theta - mB_z, \tag{1}$$

with A the vector potential and B the magnetic field, one finds [1] from $B = \nabla \times A$, $j = \nabla \times B$, and $j \times B = \nabla p$ that $H = H(G)$, $p = p(G)$, and

$$\begin{aligned} \frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{m^2 - k^2 r^2}{m^2 + k^2 r^2} \frac{\partial G}{\partial r} + \frac{m^2 + k^2 r^2}{r^2} \frac{\partial^2 G}{\partial u^2} \\ = -H \left(\frac{dH}{dG} + \frac{2mk}{m^2 + k^2 r^2} \right) - \frac{dp}{dG} (m^2 + k^2 r^2). \end{aligned} \tag{2}$$

Equation (2) can be thought of as one component of the vector equation $\nabla \times (\nabla \times A) = j$ with the equilibrium equation constraining j [$j = \sigma B + (B \times \nabla p)/B^2$, where σ is a scalar, which follows from $j \times B = \nabla p$]. Similar but more complicated systems of equations have been written to describe three-dimensional plasma equilibria (see, e.g., Ref. [2]). We use an Eulerian formulation of the plasma equilibria problem, which does not prescribe the topology of magnetic surfaces, in contrast to the Lagrangian formulation sometimes used (see, e.g., Refs. [3, 4]).

Mathematically, the plasma equilibrium equation is of the form

$$L[G] = F(G, r), \quad (3)$$

where L is a elliptic linear operator on $G(r, u)$ and F is a nonlinear functional of $G(r, u)$. In this paper, we consider solving an equation of this general form in a nearly one-dimensional case with $G(r, u) = G_0(r) + G_1(r, u)$, where $G_0(r)$ has a maximum or minimum and $G_1 \ll G_0$. In this case, any slight dependence of G on u , causes the contour plot of $G(r, u)$ to have islands, that is, the ridge (maximum) or valley (minimum) of G breaks up into regions with closed G contours.

Traditionally (see, e.g., Ref. [5]), a method we will term simple iteration has been used to solve Eq. (3). Simple iteration consists of calculating the successive iterates G^{new} from

$$L[G^{\text{new}}] = F(G^{\text{old}}, r). \quad (4)$$

The rationale for this approach has been twofold: (1) to take advantage of efficient linear solvers for the operator L , and (2) to avoid difficulties involved in taking derivatives of F . However, as we point out here there are interesting problems involving magnetic islands for which simple iteration fails. Other situations (field reversed compact torus) where simple iteration fails have been noted [6].

Simple examples are perhaps helpful. Consider the nonlinear equation $G = G^{1/2}$. A graphical illustration of simple iteration on this equation is shown in Fig. 1. The iteration clearly diverges from the root $G = 0$, which is connected with the infinite derivative of $G^{1/2}$ at $G = 0$. An analogous situation can hold for Eq. (2) as will be seen.

One can construct simple, analytically solvable cases of Eq. (2) where simple iteration fails. For example, consider the case with $k=0$, $p=0$, a strong B_z magnetic field (so that H is approximately constant), and dH/dG a linear function of G . Equation (2) is then approximately of the form

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} + \frac{1}{r^2} \frac{\partial^2 G}{\partial \theta^2} = \nabla^2 G = \alpha + \beta G. \quad (5)$$

Natural boundary conditions are regularity of G at the origin and a prescribed value of G at the outer wall, assumed to be at $r=a$. Simple iteration of Eq. (5) diverges when $|\beta| > \kappa^2/a^2$, where $\kappa = 2.405\dots$ is the first zero of $J_0(x)$.

From a practical standpoint it is helpful to understand the divergence in more detail. Let G_0 be the exact solution of Eq. (5). The error $e \equiv G - G_0$ at each step of

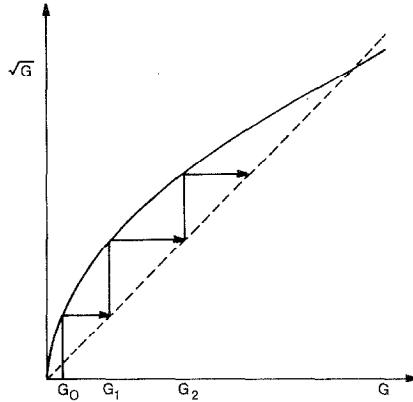


FIG. 1. Graph depicting simple iteration of the equation $G = G^{1/2}$. The iteration diverges from the root $G=0$.

the simple iteration satisfies $\nabla^2(e^{\text{new}}) = \beta(e^{\text{old}})$. By expanding in terms of eigenfunctions of the Laplacian, one sees that the error grows unless β is less than κ^2/a^2 , the smallest absolute eigenvalue of ∇^2 . For example, if the error at the first step is $e^0 = \varepsilon J_0(\kappa r/a)$, then the sequence of iterates is

$$\begin{aligned} e^0 &= \varepsilon J_0(\kappa r/a), \\ e^1 &= \varepsilon \left(-\frac{a^2}{\kappa^2} \beta \right) J_0(\kappa r/a), \\ e^2 &= \varepsilon \left(-\frac{a^2}{\kappa^2} \beta \right)^2 J_0(\kappa r/a), \\ &\vdots \end{aligned}$$

For $\beta > \kappa^2/a^2$, as assumed, the sequence diverges.

This example shows that although Eq. (5) is two dimensional, the divergent mode for the iteration is one dimensional and has no θ dependence. Thus we could have considered iteration of the one-dimensional version of Eq. (5) from the outset. In this paper we will use this approach and first study the one-dimensional version of Eq. (2). After having developed techniques to solve the one-dimensional problem, we then apply them to the two-dimensional problem.

II. FORMULATION OF A ONE-DIMENSIONAL MODEL

A case of interest that results in a considerable simplification of Eq. (2) is the case of $m=2$ islands in the Tokamak [7]. The toroidal plasma (see Fig. 2) in the Tokamak is modeled by a cylinder with identified ends as shown in Fig. 3. The

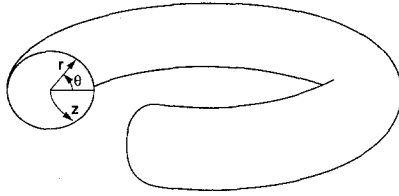


FIG. 2. Torus and cross section.

major radius of the torus is denoted by R and the minor radius by a . By periodicity, k is constrained to be n/R where n is an integer. The case $n = -1$ and $R/a = 3$ will be used as an example for numerical calculations. We assume pressure is zero and k is small. Equation (2) becomes approximately in this case

$$L[G] = \frac{1}{r}(rG')' + \frac{m^2}{r^2} \frac{\partial^2 G}{\partial u^2} = -H \left(\frac{dH}{dG} + \frac{2k}{m} \right) = F(G), \quad (6)$$

where the symbol $'$ is used to denote the radial derivative. In the Tokamak, $B_\theta \ll B_z$ and consequently $H \equiv krB_\theta - mB_z \sim -mB_0$, where B_0 is the central toroidal field ($B_z \sim B_0$). Using the notation $\sigma \equiv dH/dG$ ($\sigma = j/B$, where j is the plasma current) and nondimensionalizing ($Ga/B_0 \rightarrow G$, $r/a \rightarrow r$, $\sigma a \rightarrow \sigma$) we obtain the approximate equation

$$L[G] = 2(\sigma(G) - \frac{1}{3}). \quad (7)$$

[Equation (7) is the same as an equation given in Ref. [7], but uses a different choice of dimensionless units.]

Neglecting the effect of toroidal curvature, Tokamak equilibria are nominally one dimensional and specified by a single profile, say the current profile $\sigma(r)$. Clearly Eq. (7), with $\sigma(r)$ given, can readily be solved for $G(r)$. This can be done for any choice of m and n , providing equivalent descriptions of the underlying cylindrically symmetrical Tokamak state. To study possible bifurcations of equilibria to two-dimensional states, we need to choose m and n to match the helical state of interest and to specify $\sigma(G)$ rather than $\sigma(r)$. We first consider the one-dimensional case.

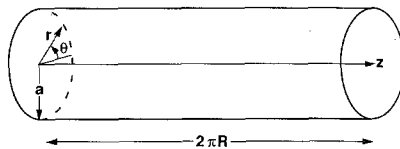


FIG. 3. Cylinder with identified ends.

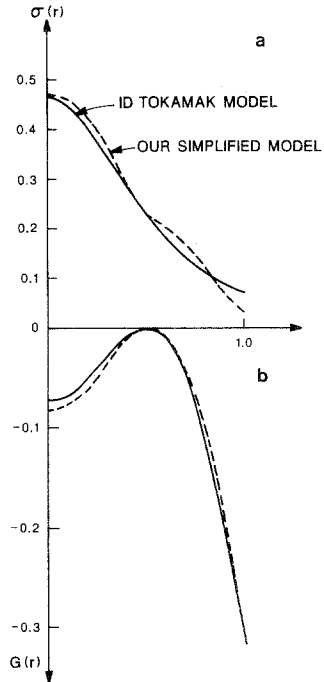


FIG. 4. A comparison of current profile $\sigma(r)$ and flux $G(r)$ for a commonly used Tokamak model and our simplified model.

In Fig. 4a is shown $\sigma(r)$ for a commonly used model of the Tokamak (see Appendix A). In order for a bifurcated equilibrium with a small magnetic island to exist the one-dimensional equilibrium must have a singular surface r_s [a maximum or minimum with $G'(r_s) = 0$]. The current profile $\sigma(r)$ given by Fig. 4a leads to such a singular surface at $r_s = 0.5$, as shown in Fig. 4b, where $G(r)$ has been obtained by solving Eq. (7). Near $r = r_s$, $G \sim G_s + G_s'' x^2/2$, where $x \equiv r - r_s$, since $G'_s = 0$. However, $\sigma \sim \sigma_s + \sigma'_s x$, without a flat region near $r = r_s$ (see Fig. 4a), so that $\sigma(G)$ must be of the form $\sigma(G) \sim \sigma_s + \alpha(G_s - G)^{1/2}$. To simplify the problem we assume this form everywhere, although with different coefficients in the region to the left (region I) and right (region II) of the singular surface. This constitutes the model that we use subsequently, namely,

$$\sigma(G) = \sigma_s \begin{cases} +\alpha(G_s - G)^{1/2}, & \text{region I,} \\ +\beta(G_s - G)^{1/2}, & \text{region II.} \end{cases} \quad (8)$$

Summarizing, our one-dimensional model is as follows:

$$L[G] = \frac{1}{r} (rG')' = F(G) = F_s \begin{cases} +\alpha(G_s - G)^{1/2}, & \text{region I,} \\ +\beta(G_s - G)^{1/2}, & \text{region II,} \end{cases} \quad (9)$$

where $F_s = -0.2066$, $\alpha = 5.9$, and $\beta = -2.5$ (see Appendix B).

The boundary conditions that go along with Eq. (9) are

$$\begin{aligned} G'(0) &= 0 \\ G(r_s) &= G_s \\ G'(r_s) &= 0, \end{aligned} \tag{10}$$

where the singular surface location r_s (the location of the maximum of G) has been introduced as an additional variable.

III. NUMERICAL METHODS IN ONE DIMENSION

In this section we compare use of simple iteration and Newton's method for solving Eq. (9). The first method is simple iteration. The next iterate G^{new} is computed by solving the discretized version of

$$L[G^{\text{new}}] = F(G^{\text{old}}), \tag{11}$$

where $L[G] = 1/r(rG)'$ for our model Tokamak problem. Equation (11) is first solved as a standard boundary value problem with the two boundary conditions $G^{\text{new}}(0) = 0$ and $G^{\text{new}}(1) = 0$. A constant [equal to $G_s - G^{\text{new}}(r_s)$] is then added to this solution to obtain a solution with $G^{\text{new}}(r_s) = G_s$ [since a solution of Eq. (11) remains a solution when a constant is added to it]. Table I shows successive maximum absolute values of the residuals

$$R \equiv F(G^{\text{new}}) - L[G^{\text{new}}]$$

obtained using simple iteration. As seen, the method does not converge. The initial guess for this method and the other methods to be discussed was a quadratic or quartic expression that had the same qualitative form as the exact solution shown in Fig. 4 and reproduced the exact solution values of $G(0)$, $G(r_s)$, and $G(1)$.

TABLE I
Sequence of Residuals for 1D Model

Iteration	Simple	Newton
0	0.235	0.235
1	0.695e-1	0.347e-1
2	0.125	0.802e-3
3	0.258	0.313e-5
4	0.585	0.582e-9

The second numerical method is Newton's method. The change in G , defined by

$$\delta G \equiv G^{\text{new}} - G^{\text{old}}$$

is required to satisfy the equation

$$L[\delta G] - \frac{dF}{dG}(G^{\text{old}}) \delta G = R^{\text{old}} \equiv F(G^{\text{old}}) - L[G^{\text{old}}]. \quad (12)$$

The boundary conditions, given by Eq. (10), involve the auxiliary variable r_s , which is defined by the condition $G'(r_s) = 0$. We formally solve this equation to eliminate r_s and obtain the boundary conditions

$$G'(0) = 0, \quad (13a)$$

$$G[(G')^{-1}(0)] = G_s. \quad (13b)$$

Applying Newton's method, we obtain

$$\delta G'(0) = -G'^{\text{old}}(0)$$

$$\delta G[(G'^{\text{old}})^{-1}(0)] + G'^{\text{old}}[(G'^{\text{old}})^{-1}(0)] \delta[(G')^{-1}(0)] = G_s - G^{\text{old}}[(G'^{\text{old}})^{-1}(0)].$$

Because $G'^{\text{old}}[(G'^{\text{old}})^{-1}(0)] \equiv 0$, the latter equation above may be written as

$$\delta G(r_s^{\text{old}}) = G_s - G^{\text{old}}(r_s^{\text{old}}), \quad (14)$$

where r_s^{old} is $(G'^{\text{old}})^{-1}(0)$.

Difficulties occur because of the square root in the definition of $F(G)$. First, the argument can be negative since Eq. (14) is only a linear approximation to the condition $\max(G^{\text{old}} + \delta G) = G_s$. This is easily taken care of by extending the definition of $F(G)$ to the case $G > G_s$. Such an extension occurs naturally in the two-dimensional case. Second, because of the square root, the derivative in Newton's method is unbounded. One must design a special one-dimensional code to handle this singularity.

The sequence of residuals for Newton's method are shown in Table I. The number of r mesh points for all one-dimensional calculations was 1000.

IV. ANALYSIS OF THE ONE-DIMENSIONAL CASE

To analyze the behavior under iteration of the one-dimensional version of Eq. (3), we consider the more general equation

$$L[\delta G] - f \frac{\partial F}{\partial G} \delta G = R, \quad (15)$$

where f is a numerical parameter such that $f = 0$ gives simple iteration and $f = 1$ Newton's method. Now consider $G(r)$ to be very close to the solution $G_0(r)$

and let $G(r) = G_0(r) + g(r)$. (Whether Eq. (15) has a solution was originally addressed by us in an appendix of a longer version of this paper. It has recently been shown (14) that equations of a type more general than Eq. (15) have unique solutions.) Linearizing the residual R , we obtain

$$R \simeq \frac{\partial F}{\partial G}(G_0) g - L[g], \quad (16)$$

using the fact that G_0 satisfies $F(G_0) = L[G_0]$. Substituting Eq. (16) into Eq. (15), we can write

$$L[g^{\text{new}}] - f \frac{\partial F}{\partial G} g^{\text{new}} = \frac{\partial F}{\partial G} (1-f) g^{\text{old}}, \quad (17)$$

where $g^{\text{new}} \equiv g + \delta g$ and $g^{\text{old}} \equiv g$. The convergence or divergence of Eq. (17) under iteration is governed by the largest absolute eigenvalue $|\lambda|$ of the linear problem defined by the equation

$$\lambda \left(L[g] - f \frac{\partial F}{\partial G} g \right) = \frac{\partial F}{\partial G} (1-f) g, \quad (18)$$

and in fact for $|\lambda| < 1$ the k th iterate is asymptotically given by

$$G^{(k)} \simeq G_0 + \lambda^k g \quad (19)$$

for large k (in nondegenerate cases), where g satisfies Eq. (18) (and does not change with k).

An eigenvalue λ of Eq. (18) is related to an eigenvalue λ_0 of the simpler problem

$$\lambda_0 L[g] = \frac{\partial F}{\partial G} g. \quad (20)$$

This relation is

$$\lambda_0 = \frac{\lambda}{1 - f + \lambda f}, \quad (21)$$

or

$$\lambda = \frac{\lambda_0(1-f)}{1 - \lambda_0 f}. \quad (22)$$

Note that the convergence factor λ is given by $\lambda = \lambda_0$ for simple iteration and $\lambda = 0$ for Newton's method. Since $d\lambda/d\lambda_0$ is always nonnegative for $f \leq 1$, eigenvalues are not reordered by Eqs. (21) and (22) and the largest eigenvalues λ and λ_0 satisfy Eqs. (21) and (22).

The boundary conditions for Eq. (20) or (18) are as given in Section III, Eq. (10). In practice Eq. (20) is solved as a standard boundary value problem with $g(1)$

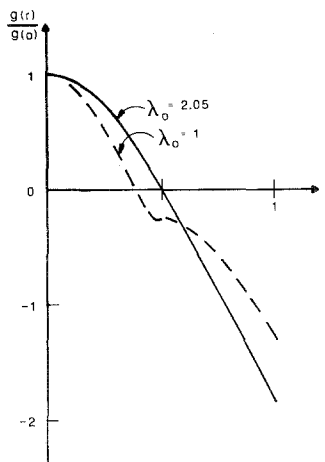


FIG. 5. Solution of the eigenvalue problem describing simple iteration. The eigenvalue λ_0 (convergence factor) is 2.05 corresponding to divergence; the form of the divergent mode is given by $g(r)$. With $\lambda_0 = 1$, the solution of the differential equation does not satisfy the boundary condition $g(r_s) = 0$.

specified. Then λ_0 is varied to satisfy $g(r_s) = 0$. In Fig. 5 we show the largest eigenvalue solution g of Eq. (20) for our model Tokamak problem. The eigenvalue λ_0 is found to have the value $\lambda_0 = 2.05$.

The requirement $\lambda^2 < 1$ is, from Eq. (22),

$$f > \frac{1}{2} + \frac{1}{2\lambda_0} \quad (23)$$

or $f > 0.74$ in this case. We find that this is the limiting value of f for convergence observed numerically. In Table II are shown the sequence of residuals for Eq. (15) with $f = 0.8$. As seen, the absolute value of the convergence factor is 0.64. This result agrees with Eq. (22), which predicts $\lambda = -0.64$ for $f = 0.8$ and $\lambda_0 = 2.05$.

Table II
Sequence of Residuals, 1D Model Using Partial Newton ($f = 0.8$)

Iteration n	R_n	R_{n+1}/R_n
0	0.235	0.115
1	$0.271e-1$	0.097
2	$0.262e-2$	0.508
3	$0.133e-2$	0.634
4	$0.843e-3$	0.631
5	$0.532e-3$	0.643
6	$0.342e-3$	0.641
7	$0.220e-3$	0.640
8	$0.141e-3$	0.640

We observe from Eq. (22) that unless f is exactly 1, λ will have some nonzero value and we therefore have *linear* convergence. Only the case $f=1$ (Newton's method) gives *superlinear* convergence, as is known in a more general context (see, e.g., Ref. [8]).

The general form of the operator L in the one-dimensional case is given by

$$L[g] = \frac{1}{U} (Ug')' \quad (24)$$

with

$$U \equiv \frac{r}{m^2 + k^2 r^2}.$$

We shall show in Appendix C that in this case, simple iteration will be unstable whenever, $\partial F/\partial G < 0$, $kH/m > 0$ in $(0, r_s)$, and $m^2 > 0.236 k^2 r_s^2$. For the Tokamak equilibria these conditions are satisfied and we expect that simple iteration will always be unstable toward an approximately one-dimensional mode.

V. SOLUTIONS IN TWO DIMENSIONS

The previous discussion leading to Eq. (9) was based on consideration of one-dimensional equilibria. The physical problem of interest is the computation of a nearby two-dimensional equilibria having a small magnetic island. The contour plot for such an equilibrium is shown in Fig. 6. There are now three flux regions because of the more complicated topography possible in two dimensions. These regions will be defined as the axis region (region I), the wall region (region II), and the island region (region III), as illustrated in Fig. 6. A reasonable physical approximation is

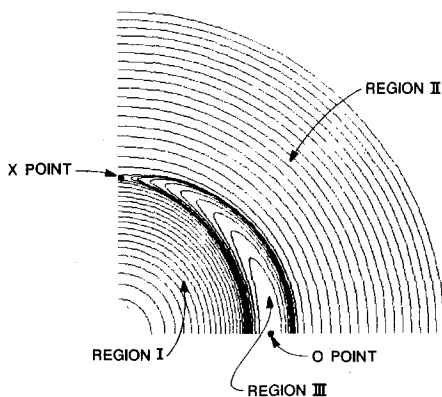


FIG. 6. Contour plot of $G(r, u)$ for a two-dimensional solution of the equilibrium equation with a magnetic island.

to assume that the form of $F(G)$ in regions I and II is given as in the one-dimensional case by Eq. (9), with singular surface values identified with X -point values. Unfortunately, because of the square root, the derivative in Newton's method is unbounded. However, in the two-dimensional case it is unphysical to have such a singularity, so we use the following replacement $\sqrt{G_x - G} \rightarrow \sqrt{G_x - G + \Delta} - \sqrt{\Delta}$ where Δ is small compared to the maximum value of $G_x - G$. (The results given have $\Delta = 10^{-4}$.) The flux in the island region, $G_O - G_X$, is small for a small island and $F(G)$ in the island region will be assumed to be representable by a two-term Taylor series as follows:

$$F(G) = F_X + \gamma(G - G_X), \quad \text{region III.} \quad (25)$$

where γ is an additional free parameter of the problem (the justification for this form involves physical considerations outside the scope of this paper). To determine γ we impose another boundary condition which is that G_O be specified. Note that by specifying $G_O \neq G_X$, we ensure that the solution will not be the one-dimensional solution. It is not, however, essential to do this and solutions can be obtained with γ specified.

The two-dimensional problem is slightly complicated by the additional parameter γ . The Newton's step of the two-dimensional problem satisfies the equations

$$L[\delta G] - \frac{\partial F}{\partial G} \delta G = R + \frac{\partial F}{\partial \gamma} \delta \gamma \quad (26)$$

where $\delta \gamma$ is the change in γ necessary to satisfy the additional boundary condition. The boundary conditions for the two-dimensional problem are

$$\begin{aligned} \frac{\partial G}{\partial r} &= 0, & r &= 0, \\ \frac{\partial G}{\partial \theta} &= 0, & \theta &= 0, \quad \theta = \frac{\pi}{m}, \quad \text{and} \quad r = 1, \end{aligned}$$

$$G_X, G_O \text{ specified.}$$

In Table III we compare the sequence of residuals for Newton's method and for simple iteration, where the $\partial F/\partial G$ term on the left-hand side of Eq. (26) is absent. The initial guess was again a quadratic or quartic form reproducing the topography of the desired solution and with G_{axis} , G_{wall} , G_O , and G_X having values close to or equal to the solution values. As seen from Table III, simple iteration diverges as expected, while Newton's method converges. A contour plot of the solution $G(r, u)$ is shown in Fig. 6. The number of mesh points for these calculations was 21 u 's and 216 r 's.

Table III
Sequence of Residuals for 2D Case with Magnetic Island

Iteration	Simple	Newton
0	0.237	0.237
1	$0.686e-1$	0.235
2	0.146	$0.301e-1$
3	0.318	$0.562e-3$
4	0.643	$0.149e-7$

VI. DISCUSSION

There are several considerations in choosing a method to solve the helically symmetric plasma equilibrium equation, Eq. (2), or possibly some analogous equation in three dimensions. Among them are convergence, rate of convergence, and solution of the linear problem.

The absolutely essential feature of any iteration scheme is that it converge rather than diverge for a sufficiently good initial guess. For calculation of equilibria with magnetic islands it is necessary to use something other than simple iteration to obtain convergence.

Rate of convergence is also very important. For the physical problem of evolving equilibria, where a sequence of nearby equilibria is calculated, Newton's method is the method of choice on the basis of rate of convergence, because Newton's method has superlinear convergence, as discussed in Section IV.

The radius of convergence is not guaranteed to be large with Newton's method and in many problems this leads to difficulties in obtaining a sufficiently accurate initial guess. Powerful techniques for handling this difficulty are continuation (homotopy) methods [10, 11]. For the problem of calculating plasma equilibria with small magnetic islands there is a natural one-dimensional approximation (the linearized equilibrium equation with derivative discontinuity across the singular surface [9] that we find to provide an adequate initial guess. Note, however, that radius of convergence is not a major issue for evolution of equilibria, again arguing for Newton's method.

The linear problem to be solved is of the form

$$\frac{\partial^2 G}{\partial r^2} + \mu(r) \frac{\partial G}{\partial r} + \nu(r) \frac{\partial^2 G}{\partial u^2} + \lambda(r, u)G = F(r, u), \quad (27)$$

where $\lambda = 0$ for the case of simple iteration. Efficient methods are readily available for solving this equation for $\lambda = \text{constant}$, but not necessarily for $\lambda = \lambda(r, u)$. We have used a direct banded matrix inversion routine based on having the number of r mesh points much larger than the number of u mesh points, which is the situation

for representation of small magnetic islands. Multigrid methods [12] for solving Eq. (27) with $\lambda = \lambda(r, u)$, or iterative methods such as the conjugate-gradient method, [13] may also be used.

It is our conclusion that Newton's method, although not traditionally used for the magnetic equilibrium problem, generally offers important advantages in rate of convergence that justify its greater complexity. Vis-à-vis simple iteration, Newton's method is necessary for certain problems, such as the problem of magnetic islands considered here. We believe that these considerations apply to the three-dimensional case as well and that Newton's method should be incorporated in the design of future three-dimensional equilibrium codes.

APPENDIX A. TOKAMAK MODEL

A commonly used tokamak model [7] assumes a q profile of the form

$$q(r) \equiv \frac{rB_z(r)}{RB_\theta(r)} = C \left(1 + \frac{r^2}{r_0^2} \right) \quad (\text{A.1})$$

(q is the winding number of the magnetic field-- q = number of toroidal transits of a magnetic field line in one poloidal transit, also known as the safety factor). Our model calculations assume Eq. (A.1) with $r_0 = 0.8a$. The condition for a singular surface at radius r_s is $nq + m = 0$, which can be solved to find the constant C ,

$$C = \frac{-m}{n[1 + (r_s/r_0)^2]}. \quad (\text{A.2})$$

Note that the condition $nq(r_s) + m = 0$ is equivalent to $G'(r_s) = 0$, with G defined by Eq. (1).

It is necessary to find the current profile $\sigma(r)$ associated with given q profile. The one-dimensional equilibrium relations from $\nabla \times B = \sigma B$ are

$$\begin{aligned} B'_z &= -\sigma B_\theta, \\ B'_\theta + \frac{B_\theta}{r} &= \sigma B_z. \end{aligned} \quad (\text{A.3})$$

Equations (A.3) and (A.1) allow one to derive the desired relationship:

$$\sigma = \frac{1}{R} \frac{2q - q'r}{q^2 + r^2/R^2}. \quad (\text{A.4})$$

APPENDIX B

In this appendix, we derive a model one-dimensional problem. Consider Eq. (7) with $\sigma(G)$ given by Eq. (8) in region I. By the transformation $G - G_s \rightarrow G$, we can assume $G_s = 0$. Three boundary conditions are

$$\begin{aligned} G'(0) &= 0, \\ G(r_s) &= 0, \\ G'(r_s) &= 0. \end{aligned}$$

Since Eq. (7) is second order and requires two boundary conditions, the additional condition specifies α , if we image σ_s to be fixed. With $r_s = 0.5$ and $\sigma_s = 0.23$ (to match Fig. 4a) the solution turns out to have $\alpha = 5.9$ and results in the dashed curves shown in region I of Fig. 4a.

In region II, two boundary conditions are

$$\begin{aligned} G(r_s) &= 0, \\ G'(r_s) &= 0. \end{aligned}$$

Specifying σ_s to match region I, β is still a free parameter. The value of β may be fixed by requiring that $G(1)$ match the value given by Fig. 4b. The solution obtained this way turns out to have $\beta = -2.5$ and gives the dashed curve shown in regions II of Fig. 4. Note that because $\beta \neq -\alpha$, σ' has a discontinuity at $r = r_s$. Choosing $\beta = -\alpha$ eliminates this discontinuity but results in poorer overall fits to $\sigma(r)$ and $G(r)$.

APPENDIX C. INSTABILITY OF SIMPLE ITERATION

In this appendix, we show that simple iteration will never converge when applied to Eq. (18), at least in the Tokamak case ($m^2 \gg k^2 r_s^2$, F given by Eq. (9)).

Let us reiterate the problem. We are interested in the possibility of finding real solutions to the nonlinear equations

$$\begin{aligned} L[G] &\equiv \frac{1}{U} (UG')' = F(G, r) = -H \left[\frac{dH}{dG} + \frac{2mk}{m^2 + k^2 r^2} \right], \\ G'(0) &= 0, \\ G(r_s) &= G_s, \\ G'(r_s) &= 0, \quad \text{where } r_s \text{ is the first zero of } G' \text{ with } r_s > 0, \end{aligned} \tag{C.1}$$

by simple iterations of the form

$$L[g^{\text{new}}] - f \frac{dF}{dG} g^{\text{new}} = \frac{\partial F}{\partial G} (1 - f) g^{\text{old}}$$

with $0 \leq f < 1$. We have shown in the body of this paper that the iteration converges or diverges depending upon whether or not the largest absolute eigenvalue $|\lambda|$ of

$$\begin{aligned} \lambda L[g] &= \frac{\partial F}{\partial G} g, \\ g'(0) &= 0 \\ g(r_s) &= 0 \end{aligned} \tag{C.2}$$

is less than 1, assuming that (C.1) has a solution. Since any solution is determined only to an additive constant, we can assume that $G(r_s) = G_s = 0$. Without loss of generality, G may be assumed negative, except at r_s , where it is zero, and since G' is of one sign on $(0, r_s)$, we must have $G' > 0$ on $(0, r_s)$ (the physically irrelevant transformation $k \rightarrow -k$, $m \rightarrow -m$ changes the sign of G). Recall that $U = r/(m^2 + k^2 r^2)$. Define the inner product of functions on $[0, r_s]$ by

$$(v, w) = \int_0^{r_s} vw U dr.$$

Let $h = G'$ in what follows.

We know that simple iteration is unstable if and only if solutions to

$$\begin{aligned} L[G] &= F(G, r), \\ \lambda L[g] &= \frac{\partial F}{\partial G} g, \end{aligned} \tag{C.3}$$

which satisfy

$$\begin{aligned} h &> 0 \quad \text{in} \quad (0, r_s), \\ h(r_s) &= g(r_s) = 0, \end{aligned} \tag{C.4}$$

have $|\lambda| \geq 1$.

We can prove the following theorem, which depends upon a standard lemma that is established after the proof of the theorem.

THEOREM. *We have*

$$\frac{\lambda - 1}{\lambda} \int_0^{r_s} hg \frac{\partial F}{\partial G} U dr = \int_0^{r_s} hg \left(\frac{U'}{U}\right)' U dr - \frac{2k}{m} \int_0^{r_s} H \frac{2rm^2k^2}{(m^2 + k^2r^2)^2} gU dr, \tag{C.5}$$

where G, g , and λ satisfy (C.3) and $h = G'$.

Proof. Since

$$\frac{2mk}{m^2 + k^2r^2} = \frac{2k}{m} - \frac{2k}{m} \left(\frac{k^2r^2}{m^2 + k^2r^2} \right),$$

we can write

$$F(G, r) = F_1(G) + \frac{2k}{m} H(k^2rU)$$

where

$$F_1(G) = -H \left[\frac{dH}{dG} + \frac{2k}{m} \right].$$

Equations (C.3) can be rewritten as

$$\begin{aligned} \frac{d}{dr} (L[G]) &= \frac{dF}{dr}, \\ \lambda h L[g] &= \left(\frac{dF}{dr} - \frac{\partial F}{\partial r} \right) g. \end{aligned} \tag{C.6}$$

Now note that

$$L \frac{d}{dr} = \frac{d}{dr} L - \left(\frac{U'}{U} \right) \frac{d}{dr}.$$

Thus, we have

$$\begin{aligned} L[h] + \left(\frac{U'}{U} \right)' h &= \frac{dF}{dr}, \\ \lambda h L[g] &= \left(\frac{dF}{dr} - \frac{\partial F}{\partial r} \right) g. \end{aligned} \tag{C.7}$$

If we multiply the first equation in (C.7) by g , we have

$$\begin{aligned} gL[h] + \left(\frac{U'}{U} \right)' hg &= \frac{dF}{dr} g \\ \lambda h L[g] &= \left(\frac{dF}{dr} - \frac{\partial F}{\partial r} \right) g. \end{aligned} \tag{C.8}$$

Integrating with respect to $U \, dr$ gives

$$(g, L[h]) + \left(\left(\frac{U'}{U} \right)', hg \right) = \lambda (h, L[g]) + \left(\frac{\partial F}{\partial r}, g \right). \tag{C.9}$$

By Lemma 1, L is self-adjoint, so we have

$$(\lambda - 1)(L[g], h) = \left(\left(\frac{U'}{U} \right)', hg \right) - \left(\frac{\partial F}{\partial r}, g \right). \tag{C.10}$$

Using the second equation in (C.3) gives

$$\begin{aligned} \frac{\lambda - 1}{\lambda} \left(\frac{\partial F}{\partial G} g, h \right) &= \left(\left(\frac{U'}{U} \right)', hg \right) - \left(\frac{\partial F}{\partial r}, g \right) \\ &= \left(\left(\frac{U'}{U} \right)', hg \right) - \frac{2k}{m} (H(k^2 r U)', g). \end{aligned} \tag{C.11}$$

This proves the theorem. A corollary of this theorem establishes the instability of simple iteration whenever $m^2 > 0.236 k^2 r_s^2$, $\partial F/\partial G < 0$, and $(2k/m)H \geq 0$. Again, the necessary lemmas will be delayed until after the proof of the corollary.

COROLLARY. *Simple iteration is unstable when $(2k/m)H \geq 0$, $\partial F/\partial G < 0$, and $m^2 > (\sqrt{5} - 2) k^2 r_s^2$.*

Proof. Recall that $h \geq 0$. Note that

$$\left(\frac{U'}{U} \right)' = \frac{5k^4 r^4 - (m^2 + 2k^2 r^2)^2}{r^2 (m^2 + k^2 r^2)^2}. \tag{C.12}$$

If $m^2 > (\sqrt{5} - 2) k^2 r_s^2 \geq (\sqrt{5} - 2) k^2 r^2$, then $m^2 + 2k^2 r^2 > \sqrt{5} k^2 r^2$ and then $(U'/U)'$ is always negative.

Since $-L$ is symmetric and positive definite (see Lemma 1), the eigenvalues λ in the second equation in (C.6) are positive (see Lemma 2). Since the Green's function of $-L$ is pointwise positive (see Lemma 3), the eigenfunction g belonging to the largest eigenvalue is pointwise positive (see Lemma 4). Thus, the right-hand side of Eq. (C.5) is negative. The integral on the left-hand side of (C.5) is negative, so we must have $\lambda > 1$. This establishes the corollary.

Remark. This is clearly the case for the Tokamak (see Section II).

Now we prove the necessary lemmas.

LEMMA 1. *The operator L is negative definite and self-adjoint with respect to the inner product (\cdot, \cdot) on the space of functions which vanish at r_s .*

Proof. Note that

$$(f, Lg) = \int_0^{r_s} f \frac{1}{U} (Ug')' U dr = fUg'|_0^{r_s} - \int_0^{r_s} f' Ug' dr = -(f', g').$$

Thus we clearly have $(Lf, g) = (f, Lg)$ and if $(f, Lf) = 0$, then $f' = 0$ so $f \equiv 0$ since $f(r_s) = 0$.

LEMMA 2. *If $w(r)$ is positive, the eigenvalues λ in*

$$-\lambda Lg = w(r) g \tag{C.13}$$

are positive.

Proof. Since

$$-\lambda \frac{1}{\sqrt{w}} L \frac{1}{\sqrt{w}} \sqrt{w} g = \sqrt{w} g$$

and $-(1/\sqrt{w})L(1/\sqrt{w})$ is a symmetric positive-definite operator if $-L$ is, the eigenvalues in (C.13) are positive.

LEMMA 3. *The Green's function of $-L$ is pointwise positive on $[0, r_s]$.*

Proof. The Green's function is seen to be

$$G(s, r) = \begin{cases} U(s) \int_r^{r_s} 1/U dx, & r_s \geq r > s, \\ U(s) \int_s^{r_s} 1/U dx, & s \geq r > 0 \end{cases}$$

which is easily seen to be positive.

LEMMA 4. *The eigenfunction g corresponding to the largest eigenvalue of (C.13) is pointwise positive.*

Proof. Clearly the Green's function \mathbf{H} for $-(1/\sqrt{w})L(1/\sqrt{w})$ is $\sqrt{w} \mathbf{G} \sqrt{w}$ where \mathbf{G} is the Green's function of $-L$. Let us show that the eigenfunction $\phi = \sqrt{w} g$ corresponding to the largest eigenvalue of

$$\lambda \phi(x) = \int_0^{r_s} \mathbf{H}(x, y) \phi(y) dy \equiv (\mathbf{H}\phi)(x)$$

for symmetric pointwise positive $\mathbf{H}(x, y)$ is positive. We use the fact that

$$\lambda = \max_{\|\phi\|=1} (\mathbf{H}\phi, \phi).$$

Now $\phi = \phi^+ - \phi^-$ where $\phi^+ = \max(\phi, 0)$ and $\phi^- = (-\phi)^+$. We may assume $\phi^+ \neq 0$. Clearly, $\|\phi^+ + \phi^-\| = \|\phi\|$. In addition,

$$(\mathbf{H}(\phi^+ + \phi^-), \phi^+ + \phi^-) = (\mathbf{H}\phi^+, \phi^+) + (\mathbf{H}\phi^-, \phi^+) + \mathbf{H}\phi^+, \phi^- + (\mathbf{H}\phi^-, \phi^-).$$

Since \mathbf{H} is pointwise positive, $(\mathbf{H}\phi^-, \phi^+) = (\mathbf{H}\phi^+, \phi^-) \geq 0$ and the above is larger than $(\mathbf{H}\phi, \phi)$ unless $\phi^- = 0$. Since ϕ supposedly gave the maximum, ϕ^- must be zero.

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